

Indian Statistical Institute, Bangalore Centre
B.Math. (III Year): 2008-2009
Semester I. Midsemestral examination
Introduction to Differential Equations

25.09.2008

Time: $2\frac{1}{2}$ hrs.

Max. Marks: 80

Note: The paper carries 83 marks. Any score above 80 will be taken as 80.

1. [4×7=28 marks] Prove or disprove the following.

(i) Let $p(\cdot)$ be a continuous function on $[a, b]$. Then $x(\cdot) \equiv 0$ is the only continuous solution to the problem: $x'(t) + p(t)x(t) = 0$, $a < t < b$, with $\lim_{t \uparrow b} x(t) = 0$.

(ii) Let $p(\cdot)$ be a continuous function on $[0, \infty)$ such that $p(\cdot) \geq 1$. Then the following problem has a unique continuous solution: $x'(t) + p(t)x(t) = 0$, $t > 0$, with $\lim_{t \rightarrow \infty} x(t) = 0$.

(iii) Let A be a constant 2×2 real matrix, and $\underline{a} \in \mathbb{R}^2$. Then the function $t \mapsto e^{tA}\underline{a}$, $t \geq 0$ is the unique solution to the problem:

$$\frac{d\underline{x}(t)}{dt} = A\underline{x}(t), \quad t > 0, \quad \text{with } \underline{x}(0) = \underline{a}.$$

(iv) Let $q(\cdot)$ be a continuous function on $[0, \infty)$ such that $q(\cdot) > 0$. Then any nontrivial solution to $x''(t) + q(t)x(t) = 0$ has at most one zero on $[0, \infty)$.

2. [10 marks] Suppose a radioactive material disintegrates at a rate proportional to the amount of material present. (Take the proportionality factor to be a known constant.) Find the time required for the mass to be reduced to one half of its original value.

3. [12 marks] Find the general solution to $x''(t) - 4x(t) + 3 = 0$, $t > 0$.

4. [12 marks] Find the general solution to $x''(t) + 3x'(t) + 2x(t) = t$, $t > 0$.

5. [6+15= 21 marks] (i) Let $p(\cdot), q(\cdot)$ be continuous functions on $[0, \infty)$. Let the differential operator L be given by

$$Lu(t) = u''(t) + p(t)u'(t) + q(t)u(t), \quad t > 0$$

whenever the r.h.s. makes sense. State clearly what is meant by Green's function of L for the initial value problem on $[0, \infty)$.

(ii) Show that Green's function in (i) is unique.